

Bi-Hamiltonian ODEs with matrix variables

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Abstract. We consider a special class of linear and quadratic Poisson brackets related to ODE systems with matrix variables. We investigate general properties of such brackets, present an example of a compatible pair of quadratic and linear brackets and found the corresponding hierarchy of integrable models, which generalizes the two-component Manakov's matrix system in the case of arbitrary number of matrices.

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1 Introduction

We consider ODE systems of the form

$$\frac{dx_\alpha}{dt} = F_\alpha(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_N), \quad (1.1)$$

where x_i are $m \times m$ -matrices and F_α are (non-commutative) polynomials. There exist systems (1.1) integrable for any m . For example, the system

$$u_t = u^2 v - v u^2, \quad v_t = 0 \quad (1.2)$$

is integrable by the Inverse Scattering Method for any size m of matrices u and v . If u is a matrix such that $u^T = -u$, and v is a constant diagonal matrix, then (1.2) is equivalent to the m -dimensional Euler top. The integrability of this model was established by S.V. Manakov in 1976 ([8]).

In this paper we construct an integrable generalization of system (1.2) to the case of arbitrary N using the bi-Hamiltonian approach [1]. This approach is based on the notion of a pair of compatible Poisson brackets. Two Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are said to be compatible if

$$\{\cdot, \cdot\}_\lambda = \{\cdot, \cdot\}_1 + \lambda \{\cdot, \cdot\}_2 \quad (1.3)$$

is a Poisson bracket for any constant λ .

If the bracket (1.3) is degenerate, then a hierarchy of integrable Hamiltonian ODE systems can be constructed via the following

Theorem 1 ([2, 3]). Let

$$C(\lambda) = C_0 + \lambda C_1 + \lambda^2 C_2 + \dots, \quad \bar{C}(\lambda) = \bar{C}_0 + \lambda \bar{C}_1 + \lambda^2 \bar{C}_2 + \dots,$$

be Taylor expansion of any two Casimir functions for the bracket $\{\cdot, \cdot\}_\lambda$. Then the coefficients C_i, \bar{C}_j are pairwise commuting with respect to both brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$.

In the opposite case when, say, the bracket $\{\cdot, \cdot\}_1$ is nondegenerate, there is another way to construct an integrable hierarchy. The ratio $R = \Pi_2 \Pi_1^{-1}$, where Π_i is the Poisson tensor for $\{\cdot, \cdot\}_i$ defines a so-called recursion operator, whose spectrum provides the set of functions in involution with respect to both brackets. In this case the formula $\Pi_k = R^k \Pi_1$ gives us an infinite sequence of pairwise compatible Poisson brackets.

For an important class of Poisson brackets related to systems (1.1) the corresponding Hamiltonian operator can be expressed in terms of left and right multiplication operators given by polynomials in x_1, \dots, x_N [7]. Such brackets possess the following two properties:

- they are GL_m -adjoint invariant;

- the bracket between traces of any two matrix polynomials $P_i(x_1, \dots, x_N)$, $i = 1, 2$ is a trace of some other matrix polynomial P_3 .

Such brackets we shall call *non-abelian Poisson brackets*

In this paper we consider compatible pairs of non-abelian Poisson brackets, where the bracket $\{\cdot, \cdot\}_1$ is linear and $\{\cdot, \cdot\}_2$ is quadratic.

2 Non-abelian Poisson brackets

We consider Poisson brackets of the following form:

$$\{x_{i,\alpha}^j, x_{i',\beta}^{j'}\} = b_{\alpha,\beta}^\gamma x_{i,\gamma}^{j'} \delta_{i'}^j - b_{\beta,\alpha}^\gamma x_{i',\gamma}^j \delta_i^{j'}, \quad (2.4)$$

and

$$\{x_{i,\alpha}^j, x_{i',\beta}^{j'}\} = r_{\alpha\beta}^{\gamma\epsilon} x_{i,\gamma}^{j'} x_{i',\epsilon}^j + a_{\alpha\beta}^{\gamma\epsilon} x_{i,\gamma}^k x_{i',\epsilon}^{j'} \delta_{i'}^j - a_{\beta\alpha}^{\gamma\epsilon} x_{i',\gamma}^k x_{i,\epsilon}^j \delta_i^{j'}, \quad (2.5)$$

where $x_{i,\alpha}^j$ are entries of the matrix x_α and δ_i^j is the Kronecker delta. The summation with respect to repeated indexes is assumed. Here and in the sequel we use Latin indexes for the entries of matrices. They vary from 1 to m . The Greek indexes varying from 1 to N are used for numbering of matrices.

Theorem 2. Brackets of the form (2.4) and (2.5) are both invariant with respect to GL_m -action $x_\alpha \rightarrow ux_\alpha u^{-1}$, where $u \in GL_m$. Moreover, these brackets satisfy the following property: the bracket between traces of any two matrix polynomials is a trace of a matrix polynomial. Any linear (respectively quadratic) Poisson bracket satisfying these two properties has the form (2.4) (respectively (2.5)).

There are a lot of publications devoted to quadratic Poisson brackets that appeared in the classical version of Inverse Scattering Method [6]. However, these brackets do not satisfy the properties of Theorem 2.

Theorem 3. 1) Formula (2.4) defines a Poisson bracket iff

$$b_{\alpha\beta}^\mu b_{\mu\gamma}^\sigma = b_{\alpha\mu}^\sigma b_{\beta\gamma}^\mu; \quad (2.6)$$

2) Formula (2.5) define a Poisson bracket iff the following relations hold:

$$r_{\alpha\beta}^{\sigma\epsilon} = -r_{\beta\alpha}^{\epsilon\sigma}, \quad (2.7)$$

$$r_{\alpha\beta}^{\lambda\sigma} r_{\sigma\tau}^{\mu\nu} + r_{\beta\tau}^{\mu\sigma} r_{\sigma\alpha}^{\nu\lambda} + r_{\tau\alpha}^{\nu\sigma} r_{\sigma\beta}^{\lambda\mu} = 0, \quad (2.8)$$

$$a_{\alpha\beta}^{\sigma\lambda} a_{\tau\sigma}^{\mu\nu} = a_{\tau\alpha}^{\mu\sigma} a_{\sigma\beta}^{\nu\lambda}, \quad (2.9)$$

$$a_{\alpha\beta}^{\sigma\lambda} a_{\sigma\tau}^{\mu\nu} = a_{\alpha\beta}^{\mu\sigma} r_{\tau\sigma}^{\lambda\nu} + a_{\alpha\sigma}^{\mu\nu} r_{\beta\tau}^{\sigma\lambda}. \quad (2.10)$$

and

$$a_{\alpha\beta}^{\lambda\sigma} a_{\tau\sigma}^{\mu\nu} = a_{\alpha\beta}^{\sigma\nu} r_{\sigma\tau}^{\lambda\mu} + a_{\sigma\beta}^{\mu\nu} r_{\tau\alpha}^{\sigma\lambda}. \quad (2.11)$$

Formula (2.6) mean that $b_{\alpha\beta}^\sigma$ are structure constants of an associative algebra. Similar Poisson brackets were considered in ([9],[15]). Let us consider the quadratic Poisson brackets (2.5).

Under change of basis $x_\alpha \rightarrow g_\alpha^\beta x_\beta$ the constants are transformed in a standard way:

$$r_{\alpha\beta}^{\gamma\sigma} \rightarrow g_\alpha^\lambda g_\beta^\mu h_\nu^\gamma h_\epsilon^\sigma r_{\lambda\mu}^{\nu\epsilon}, \quad a_{\alpha\beta}^{\gamma\sigma} \rightarrow g_\alpha^\lambda g_\beta^\mu h_\nu^\gamma h_\epsilon^\sigma a_{\lambda\mu}^{\nu\epsilon}, \quad (2.12)$$

Here $g_\alpha^\beta h_\beta^\gamma = \delta_\alpha^\gamma$.

The system of identities (2.7)-(2.11) admits the following discrete involution:

$$r_{\alpha\beta}^{\gamma\sigma} \rightarrow r_{\beta\alpha}^{\gamma\sigma}, \quad a_{\alpha\beta}^{\gamma\sigma} \rightarrow a_{\beta\alpha}^{\sigma\gamma}. \quad (2.13)$$

This involution corresponds to the matrix transposition $x_\alpha \rightarrow x_\alpha^T$. Two brackets related by (2.12),(2.13) are called *equivalent*.

There exists one more discrete involution:

$$r_{\alpha\beta}^{\gamma\sigma} \rightarrow r_{\gamma\sigma}^{\alpha\beta}, \quad a_{\alpha\beta}^{\gamma\sigma} \rightarrow a_{\gamma\sigma}^{\alpha\beta}. \quad (2.14)$$

The brackets related by (2.14) in principle may have different properties, for example different sets of Casimir functions.

Let V be a linear space with a basis v_α , $\alpha = 1, \dots, N$. Define linear operators r , a on the space $V \otimes V$ by

$$rv_\alpha \otimes v_\beta = r_{\alpha\beta}^{\sigma\epsilon} v_\sigma \otimes v_\epsilon, \quad av_\alpha \otimes v_\beta = a_{\alpha\beta}^{\sigma\epsilon} v_\sigma \otimes v_\epsilon.$$

Then the identities (2.7)-(2.11) can be rewritten in the following form:

$$r^{12} = -r^{21}, \quad r^{23}r^{12} + r^{31}r^{23} + r^{12}r^{31} = 0,$$

$$a^{12}a^{31} = a^{31}a^{12},$$

$$\sigma^{23}a^{13}a^{12} = a^{12}r^{23} - r^{23}a^{12},$$

$$a^{32}a^{12} = r^{13}a^{12} - a^{32}r^{13}.$$

Here all operators act in $V \otimes V \otimes V$, by σ^{ij} we mean the transposition of i -th and j -th component of the tensor product and a^{ij} , r^{ij} mean operators a , r acting in the product of the i -th and j -th components.

The involution (2.14) corresponds to $a \rightarrow a^*$, $r \rightarrow r^*$, where a^* , r^* act in dual space V^* ; (2.13) corresponds to $a \rightarrow \sigma a \sigma$, $r \rightarrow \sigma r$ where σ acts by the permutation of vector spaces in the space $V \otimes V$. The equivalence transformation (2.12) corresponds to $a \rightarrow GaG^{-1}$, $r \rightarrow GrG^{-1}$, where $G = g \otimes g$ and $g \in GL(V)$.

There is a subclass of brackets (2.5) that corresponds to the tensor a equals to 0. Relations (2.7), (2.8) mean that r is a constant solution of the associative Yang-Baxter equation ([5],[17]). Such tensors r can be constructed in the following algebraic way.

An *anti-Frobenius algebra* is an associative algebra \mathcal{A} (not necessarily with unity) with non-degenerate anti-symmetric bilinear form $(\ , \)$ satisfying the following relation

$$(x, yz) + (y, zx) + (z, xy) = 0 \quad (2.15)$$

for all $x, y, z \in \mathcal{A}$. In other words the form $(\ , \)$ defines a cyclic 1-cocycle on \mathcal{A} .

Theorem 4. There exists one-to-one correspondence between solutions of (2.7), (2.8) up to equivalence and exact representations of anti-Frobenius algebras up to isomorphism.

Proof. Tensor r can be written as $r_{kl}^{ij} = \sum_{\alpha, \beta=1}^p g^{\alpha\beta} y_{k,\alpha}^i y_{l,\beta}^j$, where $g^{\alpha\beta} = -g^{\beta\alpha}$, the matrix $G = (g^{\alpha\beta})$ is non-degenerate and p is the smallest possible. Substituting this representation into (2.7), (2.8), we obtain that there exists a tensor $\phi_{\alpha\beta}^\gamma$ such that $y_{k,\alpha}^i y_{j,\beta}^k = \phi_{\alpha\beta}^\gamma y_{j,\gamma}^i$. Let \mathcal{A} be the associative algebra with the basis y_1, \dots, y_p and the product $y_\alpha y_\beta = \phi_{\alpha\beta}^\gamma y_\gamma$. Define the anti-symmetric bilinear form by $(y_\alpha, y_\beta) = g_{\alpha\beta}$, where $(g_{\alpha\beta}) = G^{-1}$. Then (2.7), (2.8) is equivalent to anti-Frobenius property (2.15).

Example 1 (cf. [4]). Let \mathcal{A} be associative algebra of $N \times N$ -matrices with zero N -th row, l be generic element of \mathcal{A}^* . Then $(x, y) = l([x, y])$ is a non-degenerate anti-symmetric bilinear form satisfying (2.15). Let $(x, y) = \text{trace}([x, y] k^T)$, where $k \in \mathcal{A}$. Then we can put $k_{ij} = 0, i \neq j, k_{ii} = \mu_i$, where $i, j = 1, \dots, N-1$, and $k_{iN} = 1, i = 1, \dots, N-1$. The corresponding bracket (2.5) is given by the following tensor r :

$$r_{Ni}^{ii} = -r_{iN}^{ii} = 1, \quad r_{ij}^{ij} = r_{ij}^{ji} = r_{ji}^{ii} = -r_{ij}^{ii} = \frac{1}{\mu_i - \mu_j}, \quad i \neq j, \quad i, j = 1, \dots, N-1. \quad (2.16)$$

The remaining elements of the tensor r and all elements of the tensor a supposed to be zero. Notice that this tensor is anti-symmetric with respect to involution (2.13). The bracket (2.16) is equivalent to

$$r_{\alpha\beta}^{\alpha\beta} = r_{\alpha\beta}^{\beta\alpha} = r_{\beta\alpha}^{\alpha\alpha} = -r_{\alpha\beta}^{\alpha\alpha} = \frac{1}{\lambda_\alpha - \lambda_\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, N. \quad (2.17)$$

Here $\lambda_1, \dots, \lambda_N$ are arbitrary pairwise distinct parameters. For $m = 1$ we have the following scalar Poisson bracket

$$\{x_\alpha, x_\beta\} = \frac{(x_\alpha - x_\beta)^2}{\lambda_\beta - \lambda_\alpha}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, N.$$

If N is even, then the Poisson structure (2.17) is non-degenerate, i.e. the rank of the Poisson tensor Π is equal to Nm^2 . In the odd case $\text{rank } \Pi = (N-1)m^2$.

There is the following known way (the so-called argument shift method) for constructing a linear Poisson bracket compatible with a quadratic one. A vector $\mu = (\mu_1, \dots, \mu_m)$ is said to be *admissible* if for any α, β

$$(a_{\alpha\beta}^{\sigma\epsilon} - a_{\beta\alpha}^{\epsilon\sigma} + r_{\alpha\beta}^{\sigma\epsilon}) \mu_\sigma \mu_\epsilon = 0.$$

For any admissible vector the argument shift $x_\alpha \rightarrow x_\alpha + \mu_\alpha \text{Id}$ yields a linear Poisson bracket with

$$b_{\alpha\beta}^\sigma = (a_{\alpha\beta}^{\epsilon\sigma} + a_{\alpha\beta}^{\sigma\epsilon} + r_{\alpha\beta}^{\sigma\epsilon})\mu_\epsilon,$$

compatible with the quadratic one. For Example 1 any admissible vector is proportional to $(1, 1, \dots, 1)$ and the corresponding linear bracket is trivial.

Example 2. Applying the involution (2.14) to (2.17), we get one more example with zero tensor a :

$$r_{\alpha\beta}^{\alpha\beta} = r_{\beta\alpha}^{\alpha\beta} = r_{\alpha\alpha}^{\beta\alpha} = -r_{\alpha\alpha}^{\alpha\beta} = \frac{1}{\lambda_\alpha - \lambda_\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, N. \quad (2.18)$$

It is easy to verify that in this case any vector (μ_1, \dots, μ_N) is admissible. All entries of the matrix $\sum_1^N x_\alpha$ are Casimir functions for both quadratic Poisson bracket $\{\cdot, \cdot\}_2$ of Example 2 and for the corresponding linear bracket $\{\cdot, \cdot\}_1$. Thus we can fix

$$\sum_1^N x_\alpha = C,$$

where C is a constant matrix. Hamiltonians of the hierarchy commuting with respect to both $\{\cdot, \cdot\}_2$ and $\{\cdot, \cdot\}_1$ are given by

$$\text{tr } x_\alpha^k, \quad \text{tr } x_\alpha^k \sum_{\beta \neq \alpha} \frac{x_\beta}{\lambda_\alpha - \lambda_\beta}, \quad k = 1, 2, \dots$$

The Casimir functions of $\{\cdot, \cdot\}_1$ belong to this set, which gives a constructive way to find the whole hierarchy.

The dynamical system corresponding to the simplest Hamiltonian $\text{tr } x_N$ and the Poisson structure $\{\cdot, \cdot\}_2$ has the form

$$\frac{dx_\alpha}{dt} = \frac{x_N x_\alpha - x_\alpha x_N}{\lambda_N - \lambda_\alpha}, \quad \alpha = 1, \dots, N-1.$$

The linear Casimir functions for $\{\cdot, \cdot\}_1$ are $\text{tr } x_\alpha$, where $\alpha = 1, \dots, N$. There exists the following quadratic Casimir function:

$$H = \frac{1}{2} \sum_{\alpha=1}^N \frac{1}{\mu_\alpha} \text{tr } x_\alpha^2.$$

The non-abelian system corresponding to this Hamiltonian and the Poisson bracket $\{\cdot, \cdot\}_2$ is given by

$$\frac{dx_\alpha}{dt} = \sum_{\beta \neq \alpha} \frac{x_\alpha x_\beta^2 - x_\beta^2 x_\alpha}{(\lambda_\alpha - \lambda_\beta) \mu_\beta} + \sum_{\beta \neq \alpha} \frac{x_\beta x_\alpha^2 - x_\alpha^2 x_\beta}{(\lambda_\alpha - \lambda_\beta) \mu_\alpha}. \quad (2.19)$$

The system (2.19) can be written in the following bi-Hamiltonian form:

$$\frac{d\mathbf{x}}{dt} = \{\mathbf{x}, \text{grad}(\text{tr } H)\}_2 = \{\mathbf{x}, \text{grad}(\text{tr } K)\}_1,$$

where

$$K = \frac{1}{3} \sum_{\alpha=1}^N \frac{1}{\mu_{\alpha}^2} \text{tr } x_{\alpha}^3.$$

If $N = 2$, then system (2.19) is equivalent to (1.2).

3 Conclusions and outlooks

We have proposed some examples of linear and quadratic Poisson brackets naturally related to some matrix ODE. Similar Poisson, symplectic and many other interesting algebraic structures were appeared recently in the framework of the M.Kontsevich's approach to the Non-Commutative Symplectic Geometry ([10]). We mention here the works of P.Etingof, V. Ginzburg, W. Crawley-Boevey ([13],[14]), L. Le Bryun ([12]), M. Van den Bergh ([11]) and many others on Calogero-Moser spaces, symplectic and Poisson geometry of Quiver Path algebras, necklace and double Poisson structures, Leibniz-Loday algebras, Rota-Baxter algebras etc.

A forthcoming paper [16] establishes the place of our non-abelian Poisson structures in the realm of these non-commutative algebro-geometric notions. We shall discuss the relations of our quadratic Poisson structures to the Poisson geometry of the affine variety associated with the representation spaces modulo adjoint GL -action, describe classification results for quadratic brackets for a free associative algebra case and the link with the "quadratic" double Poisson structures. We shall describe their symplectic foliations and Casimirs as well as the correspondent integrable systems.

Another interesting and intriguing problem relates to the quantization of the non-abelian brackets, their relations to different versions of dynamical Yang-Baxter equations and their generalizations, the quantization of the non-abelian integrable equations. All that is beyond the scope of the papers. We hope to return to this subject elsewhere.

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